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Behavior of solutions of a second order rational difference equation

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ABSTRACT. In this paper, we solve the difference equation

$$x_{n+1} = \frac{\alpha}{x_n x_{n-1} - 1}, \quad n = 0, 1, \dots,$$

where $\alpha > 0$ and the initial values x_{-1}, x_0 are real numbers. We find some invariant sets and discuss the global behavior of the solutions of that equation. We show that when $\alpha > \frac{2}{3\sqrt{3}}$, under certain conditions there exist solutions, that are either periodic or converging to periodic solutions. We show also the existence of dense solutions in the real line. Finally, we show that when $\alpha < \frac{2}{3\sqrt{3}}$, one of the negative equilibrium points attracts all orbits with initials outside a set of Lebesgue measure zero.

1. INTRODUCTION

In [9], Amleh et al. studied the difference equation

$$x_{n+1} = \frac{\alpha}{x_n x_{n-1} + 1}, \quad n = 0, 1, \dots, \tag{1}$$

where the α is positive and the initial conditions are nonnegative real numbers. They conjectured that every solution has a finite limit but confirmed it only when $\alpha \leq 2$.

In [8], The authors studied the difference equation

$$x_{n+1} = \frac{\alpha}{1 + x_n x_{n-1} + c x_{n-1}}, \quad n = 0, 1, \dots,$$

where the α is positive and the initial conditions are nonnegative real numbers. They conjectured that the unique positive equilibrium point is globally asymptotically stable and confirmed it only when $(\alpha - c)^2 \leq 4$.

Kulenović et al. [24], studied equation (1) and gave a unified proof for all values of α that the unique equilibrium is globally asymptotically stable.

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For more on difference equations with quadratic terms, see [1]-[14], [16], [17], [19], [21]-[25], [27]-[29].

In this paper, we study the difference equation

$$x_{n+1}=\frac{\alpha}{x_n x_{n-1}-1},\quad n=0,1,\ldots, \tag{2}$$

where $\alpha>0$ and the initial conditions are real numbers. The transformation

$$x_n=\frac{y_{n-1}}{y_n},\quad \text{with}\quad y_{-2}=1,$$

reduces the difference equation (2) into the linear third order difference equation

$$y_{n+1}+\frac{1}{\alpha}y_n-\frac{1}{\alpha}y_{n-2}=0,\quad n=0,1,\ldots. \tag{3}$$

The characteristic equation of equation (3) is

$$\lambda^3+\frac{1}{\alpha}\lambda^2-\frac{1}{\alpha}=0. \tag{4}$$

Clear that equation (4) has a positive real root λ_0 for all values of α.

Equation (4) can be written as

$$\lambda^3+\frac{1}{\alpha}\lambda^2-\frac{1}{\alpha}=(\lambda-\lambda_0)\left(\lambda^2+\Big(\lambda_0+\frac{1}{\alpha}\Big)\lambda+\lambda_0\Big(\lambda_0+\frac{1}{\alpha}\Big)\right)=0.$$

Therefore, the roots of equation (4) are

$$\lambda_0,\quad \lambda_\pm=-\frac{\lambda_0+\frac{1}{\alpha}}{2}\pm\frac{\sqrt{(\lambda_0+\frac{1}{\alpha})^2-4\lambda_0(\lambda_0+\frac{1}{\alpha})}}{2}.$$

We have the following result:

Lemma 1.1. *For equation (4), we have the following:*

1. *If $\alpha>\frac{2}{3\sqrt{3}}$, then equation (4) has one positive real root and two complex conjugate roots.*
2. *If $\alpha=\frac{2}{3\sqrt{3}}$, then equation (4) has one positive real root and a repeated negative real root.*
3. *If $\alpha<\frac{2}{3\sqrt{3}}$, then equation (4) has three real different roots, one of them is positive and two negative roots.*

Proof. It is sufficient to see that, the discriminant of the polynomial

$$p(\lambda)=\lambda^3+\frac{1}{\alpha}\lambda^2-\frac{1}{\alpha}=0$$

is

$$\triangle=-4\frac{1}{\alpha^4}+27\frac{1}{\alpha^2}.$$

□

2. Forbidden set and solution of equation (2)

As the solution of equation (2) depends on α, we shall consider the three cases given in Lemma 1.1.

Case $\alpha > \frac{2}{3\sqrt{3}}$:

When $\alpha > \frac{2}{3\sqrt{3}}$, the roots of equation (4) are

$$\lambda_0 \quad \text{and} \quad \lambda_{\pm} = -\frac{\lambda_0 + \frac{1}{\alpha}}{2} \pm i\frac{\sqrt{4\lambda_0(\lambda_0 + \frac{1}{\alpha}) - (\lambda_0 + \frac{1}{\alpha})^2}}{2}.$$

Then the solution of equation (2) is

$$x_n = \frac{c_1\lambda_0^{n-1} + (\frac{1}{\alpha\lambda_0})^{\frac{n-1}{2}}(c_2\cos(n-1)\theta + c_3\sin(n-1)\theta)}{c_1\lambda_0^n + (\frac{1}{\alpha\lambda_0})^{\frac{n}{2}}(c_2\cos n\theta + c_3\sin n\theta)}, \tag{5}$$

where

$$|\lambda_{\pm}| = \sqrt{\lambda_0\left(\lambda_0 + \frac{1}{\alpha}\right)} = \sqrt{\frac{1}{\alpha\lambda_0}},$$

$$\theta = \tan^{-1}\left(-\sqrt{\frac{3\lambda_0\alpha - 1}{\lambda_0\alpha + 1}}\right) \in \left]\frac{\pi}{2}, \pi\right[.$$

Using the initials y_{-2}, y_{-1} and y_0, the values of c_1, c_2 and c_3 are:

$$\begin{aligned} c_1 &= (y_0c_{11} + y_{-1}c_{12} + y_{-2}c_{13}),\\ c_2 &= (y_0c_{21} + y_{-1}c_{22} + y_{-2}c_{23}),\\ c_3 &= (y_0c_{31} + y_{-1}c_{32} + y_{-2}c_{33}), \end{aligned} \tag{6}$$

where

$$\begin{array}{lll} c_{11} = -\frac{1}{\Delta_1}\lambda_0\alpha\sqrt{\lambda_0\alpha}\sin\theta, & c_{12} = \frac{1}{\Delta_1}\lambda_0\alpha\sin 2\theta, & c_{13} = -\frac{1}{\Delta_1}\sqrt{\lambda_0\alpha}\sin\theta,\\ c_{21} = \frac{1}{\Delta_1}\Big(\alpha\sin 2\theta - \frac{\sqrt{\lambda_0\alpha}}{\lambda_0^2}\sin\theta\Big), & c_{22} = -\frac{1}{\Delta_1}\lambda_0\alpha\sin 2\theta, & c_{23} = -\frac{1}{\Delta_1}\sqrt{\lambda_0\alpha}\sin\theta,\\ c_{31} = \frac{1}{\Delta_1}\Big(\alpha\cos 2\theta - \frac{\sqrt{\lambda_0\alpha}}{\lambda_0^2}\cos\theta\Big), & c_{32} = \frac{1}{\Delta_1}\Big(-\lambda_0\alpha\cos 2\theta + \frac{1}{\lambda_0^2}\Big), & c_{33} = \frac{1}{\Delta_1}\Big(\sqrt{\lambda_0\alpha}\cos\theta - \frac{1}{\lambda_0}\Big), \end{array} \tag{7}$$

and

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 0\\ \frac{1}{\lambda_0} & \sqrt{\lambda_0\alpha}\cos\theta & -\sqrt{\lambda_0\alpha}\sin\theta\\ \frac{1}{\lambda_0^2} & \lambda_0\alpha\cos 2\theta & -\lambda_0\alpha\sin 2\theta \end{vmatrix}.$$

By simple calculations, we can write the solution of equation (3) as

$$y_n = \gamma_{1n}y_0 + \gamma_{2n}y_{-1} + \gamma_{3n}y_{-2},$$

where

$$\gamma_{1n} = c_{11}\lambda_0^n + c_{21}\Big(\frac{1}{\lambda_0\alpha}\Big)^{\frac{n}{2}}\cos n\theta + c_{31}\Big(\frac{1}{\lambda_0\alpha}\Big)^{\frac{n}{2}}\sin n\theta,$$

$$\gamma_{2n} = c_{12}\lambda_0^n + c_{22}\Big(\frac{1}{\lambda_0\alpha}\Big)^{\frac{n}{2}}\cos n\theta + c_{32}\Big(\frac{1}{\lambda_0\alpha}\Big)^{\frac{n}{2}}\sin n\theta \tag{8}$$

$$\gamma_{3n} = c_{13}\lambda_0^n + c_{23}\Big(\frac{1}{\lambda_0\alpha}\Big)^{\frac{n}{2}}\cos n\theta + c_{33}\Big(\frac{1}{\lambda_0\alpha}\Big)^{\frac{n}{2}}\sin n\theta$$

are such that c_{ij}, $i, j = 1, 2, 3$ are given in (7).

Case $\alpha = \frac{2}{3\sqrt{3}}$:

When $\alpha = \frac{2}{3\sqrt{3}}$, the roots of equation (4) are

$$\lambda_0 = \frac{1}{3\alpha}, \quad -\frac{2}{3\alpha}, \quad -\frac{2}{3\alpha}.$$

Then the solution of equation (2) is

$$x_n = \frac{c_1(\frac{1}{3\alpha})^{n-1} + c_2(-\frac{2}{3\alpha})^{n-1} + c_3(-\frac{2}{3\alpha})^{n-1}(n-1)}{c_1(\frac{1}{3\alpha})^n + c_2(-\frac{2}{3\alpha})^n + c_3(-\frac{2}{3\alpha})^n n}.$$

Using the initials y_{-2}, y_{-1} and y_0, the values of c_1, c_2 and c_3 in this case are:

$$\begin{aligned} c_1 &= y_0c_{11} + y_{-1}c_{12} + y_{-2}c_{13}, \\ c_2 &= y_0c_{21} + y_{-1}c_{22} + y_{-2}c_{23}, \\ c_3 &= y_0c_{31} + y_{-1}c_{32} + y_{-2}c_{33}, \end{aligned}$$

where

$$\begin{aligned} &c_{11} = \frac{1}{\Delta_2}\frac{27}{8}\alpha^3, \quad &&c_{12} = \frac{1}{\Delta_2}\frac{9}{2}\alpha^2, \quad &&c_{13} = \frac{1}{\Delta_2}\frac{3}{2}\alpha, \\ &c_{21} = \frac{1}{\Delta_2}27\alpha^3, &&c_{22} = -\frac{9}{2}\alpha^2, &&c_{23} = -\frac{1}{\Delta_2}\frac{3}{2}\alpha, \\ &c_{31} = \frac{1}{\Delta_2}\frac{81}{4}\alpha^3, &&c_{32} = \frac{1}{\Delta_2}\frac{27}{4}\alpha^2, &&c_{33} = -\frac{1}{\Delta_2}\frac{9}{2}\alpha \end{aligned} \tag{9}$$

and

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 0 \\ 3\alpha & -\frac{3}{2}\alpha & \frac{3}{2}\alpha \\ (3\alpha)^2 & (-\frac{3\alpha}{2})^2 & -2(-\frac{3\alpha}{2})^2 \end{vmatrix}.$$

By simple calculations, we can write the solution of equation (3) in this case as

$$y_n = \gamma_{1n}y_0 + \gamma_{2n}y_{-1} + \gamma_{3n}y_{-2},$$

where

$$\begin{aligned}\gamma_{1n} &= c_{11}(\tfrac{1}{3\alpha})^n + c_{21}(-\tfrac{2}{3\alpha})^n + c_{31}(-\tfrac{2}{3\alpha})^n n,\\ \gamma_{2n} &= c_{12}(\tfrac{1}{3\alpha})^n + c_{22}(-\tfrac{2}{3\alpha})^n + c_{32}(-\tfrac{2}{3\alpha})^n n,\\ \gamma_{3n} &= c_{13}(\tfrac{1}{3\alpha})^n + c_{23}(-\tfrac{2}{3\alpha})^n + c_{33}(-\tfrac{2}{3\alpha})^n n\end{aligned} \tag{10}$$

are such that c_{ij}, $i, j = 1, 2, 3$ are given in (9).

Case $\alpha < \frac{2}{3\sqrt{3}}$:

When $\alpha < \frac{2}{3\sqrt{3}}$, the roots of equation (4) are

$$\lambda_0 \quad \text{and} \quad \lambda_\pm = -\frac{\lambda_0 + \frac{1}{\alpha}}{2} \pm \frac{\sqrt{(\lambda_0 + \frac{1}{\alpha})^2 - 4\lambda_0(\lambda_0 + \frac{1}{\alpha})}}{2},$$

where

$$0 < \lambda_0 < |\lambda_+| < |\lambda_-|.$$

Then the solution of equation (2) is

$$x_n = \frac{c_1\lambda_0^{n-1} + c_2\lambda_-^{n-1} + c_3\lambda_+^{n-1}}{c_1\lambda_0^n + c_2\lambda_-^n + c_3\lambda_+^n}.$$

Using the initials y_{-2}, y_{-1} and y_0, the values of c_1, c_2 and c_3 in this case are:

$$\begin{aligned}c_1 &= y_0c_{11} + y_{-1}c_{12} + y_{-2}c_{13},\\ c_2 &= y_0c_{21} + y_{-1}c_{22} + y_{-2}c_{23},\\ c_3 &= y_0c_{31} + y_{-1}c_{32} + y_{-2}c_{33},\end{aligned}$$

where

$$\begin{aligned} c_{11} &= \frac{1}{\Delta_3}\frac{\lambda_- - \lambda_+}{\lambda_-^2\lambda_+^2}, & c_{12} &= \frac{1}{\Delta_3}\frac{-\lambda_-^2 + \lambda_+^2}{\lambda_-^2\lambda_+^2}, & c_{13} &= \frac{1}{\Delta_3}\frac{\lambda_- - \lambda_+}{\lambda_-\lambda_+},\\ c_{21} &= \frac{1}{\Delta_3}\frac{\lambda_+ - \lambda_0}{\lambda_+^2\lambda_0^2}, & c_{22} &= \frac{1}{\Delta_3}\frac{\lambda_0^2 - \lambda_+^2}{\lambda_+^2\lambda_0^2}, & c_{23} &= \frac{1}{\Delta_3}\frac{\lambda_+ - \lambda_0}{\lambda_+\lambda_0},\\ c_{31} &= \frac{1}{\Delta_3}\frac{\lambda_0 - \lambda_-}{\lambda_0^2\lambda_-^2}, & c_{32} &= \frac{1}{\Delta_3}\frac{\lambda_-^2 - \lambda_0^2}{\lambda_0^2\lambda_-^2}, & c_{33} &= \frac{1}{\Delta_3}\frac{\lambda_0 - \lambda_-}{\lambda_0\lambda_-}\end{aligned} \tag{11}$$

and

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 1\\ \frac{1}{\lambda_0} & \frac{1}{\lambda_-} & \frac{1}{\lambda_+}\\ \frac{1}{\lambda_0^2} & \frac{1}{\lambda_-^2} & \frac{1}{\lambda_+^2}\end{vmatrix}.$$

By simple calculations, we can write the solution of equation (3) in this case as

$$y_n = \gamma_{1n}y_0 + \gamma_{2n}y_{-1} + \gamma_{3n}y_{-2},$$

where

$$\begin{aligned}\gamma_{1n} &= c_{11}\lambda_0^n + c_{21}\lambda_-^n + c_{31}\lambda_+^n,\\ \gamma_{2n} &= c_{12}\lambda_0^n + c_{22}\lambda_-^n + c_{32}\lambda_+^n\\ \gamma_{3n} &= c_{13}\lambda_0^n + c_{23}\lambda_-^n + c_{33}\lambda_+^n\end{aligned} \tag{12}$$

are such that c_{ij}, $i,j = 1,2,3$ are given in (11).

Using the previous arguments we can give the form of the forbidden set in the following result.

Theorem 2.1. *The forbidden set of equation* (2) *as*

$$F = \bigcup_{n=-1}^{\infty} \{(x_0, x_{-1}) \in \mathbb{R}^2 : \frac{\gamma_{1n}}{x_0 x_{-1}} + \frac{\gamma_{2n}}{x_{-1}} + \gamma_{3n} = 0\},$$

where γ_{1n}, γ_{2n} *and* γ_{3n} *are given as follows:*

$$\begin{cases} \gamma_{1n}, \gamma_{2n} \text{ and } \gamma_{3n} \text{ are given in } (8), & \alpha > \frac{2}{3\sqrt{3}};\\ \gamma_{1n}, \gamma_{2n} \text{ and } \gamma_{3n} \text{ are given in } (10), & \alpha = \frac{2}{3\sqrt{3}};\\ \gamma_{1n}, \gamma_{2n} \text{ and } \gamma_{3n} \text{ are given in } (12), & \alpha < \frac{2}{3\sqrt{3}}.\end{cases}$$

Proof. The transformation

$$x_n = \frac{y_{n-1}}{y_n}, \quad \text{with} \quad y_{-2} = 1,$$

reduces the difference equation (2) into the linear third order difference equation

$$y_{n+1} + \frac{1}{\alpha} y_n - \frac{1}{\alpha} y_{n-2} = 0, \quad n = 0, 1, \ldots.$$

This equation has the characteristic equation

$$\lambda^3 + \frac{1}{\alpha}\lambda^2 - \frac{1}{\alpha} = 0.$$

The solution of the characteristic equation depends on the value of its discriminant according to Lemma (1.1).

When $\alpha > \frac{2}{3\sqrt{3}}$, the roots of the characteristic equation are

$$\lambda_0 \quad \text{and} \quad \lambda_{\pm} = -\frac{\lambda_0 + \frac{1}{\alpha}}{2} \pm i\frac{\sqrt{4\lambda_0(\lambda_0 + \frac{1}{\alpha}) - (\lambda_0 + \frac{1}{\alpha})^2}}{2},$$

where λ_0 is a positive real root.

Then

$$y_n = c_1\lambda_0^n + \left(\frac{1}{\alpha\lambda_0}\right)^{\frac{n}{2}} (c_2 \cos n\theta + c_3 \sin n\theta),$$

where

$$|\lambda_{\pm}| = \sqrt{\lambda_0(\lambda_0 + \frac{1}{\alpha})} = \sqrt{\frac{1}{\alpha\lambda_0}} \quad \text{and} \quad \theta = \tan^{-1}\left(-\sqrt{\frac{3\lambda_0\alpha - 1}{\lambda_0\alpha + 1}}\right) \in \left]\frac{\pi}{2}, \pi\right[.$$

Using the initials y_{-2}, y_{-1} and y_0, we can find the values of c_1, c_2 and c_3. These values are given in formulas (6). By simple calculations, we can write the solution

$$y_n = \gamma_{1n} y_0 + \gamma_{2n} y_{-1} + \gamma_{3n} y_{-2},$$

where γ_{1n}, γ_{2n} and γ_{3n} are given in formulas (8).

But as

$$y_{-1} = \frac{1}{x_{-1}}, \; y_0 = \frac{1}{x_0 x_{-1}},$$

we can write

$$y_n = \frac{\gamma_{1n}}{x_0 x_{-1}} + \frac{\gamma_{2n}}{x_{-1}} + \gamma_{3n}.$$

Using the transformation

$$x_n = \frac{y_{n-1}}{y_n}, \quad \text{with} \quad y_{-2} = 1,$$

it is clear that x_n is well-defined whenever $y_n \neq 0$, $n \geq -1$.

Therefore, the forbidden set in this case is

$$F = \bigcup_{n=-1}^{\infty} \left\{ (x_0, x_{-1}) \in \mathbb{R}^2 : \frac{\gamma_{1n}}{x_0 x_{-1}} + \frac{\gamma_{2n}}{x_{-1}} + \gamma_{3n} = 0 \right\},$$

where γ_{1n}, γ_{2n} and γ_{3n} are given in formulas (8).

When $\alpha = \frac{2}{3\sqrt{3}}$ and $\alpha < \frac{2}{3\sqrt{3}}$, the proof is similar and will be omitted. The proof is complete. □

3. INVARIANT SETS FOR EQUATION (2)

In this section, we shall give invariant sets for equation (2).

When $\alpha > \frac{2}{3\sqrt{3}}$, we can write the constant

$$c_1 = y_0 c_{11} + y_{-1} c_{12} + y_{-2} c_{13}$$

in terms of x_0 and x_{-1} as

$$c_1(x_0, x_{-1}) = \frac{1}{x_0 x_{-1}} c_{11} + \frac{1}{x_{-1}} c_{12} + c_{13}.$$

By simple calculations, we can show that if (x_0, x_{-1}) is such that $c_1(x_0, x_{-1}) = 0$, then (x_0, x_{-1}) lies on the rectangular hyperbola

$$\frac{\alpha \lambda_0}{x_0 x_{-1}} + \frac{1}{\lambda_0 x_{-1}} + 1 = 0.$$

Consider the set

$$D_1 = \left\{ (x, y) \in \mathbb{R}^2 : \frac{\alpha \lambda_0}{xy} + \frac{1}{\lambda_0 y} + 1 = 0 \right\}.$$

Theorem 3.1. *The set D_1 is an invariant for equation* (2).

Proof. Let $(x_0, x_{-1}) \in D_1$. We show that $(x_n, x_{n-1}) \in D_1$ for each $n \in N$. The proof is by induction on n. The point $(x_0, x_{-1}) \in D_1$, implies

$$\frac{\alpha\lambda_0}{x_0 x_{-1}} + \frac{1}{\lambda_0 x_{-1}} + 1 = 0.$$

Now for $n = 1$, we have

$$\begin{aligned}\frac{\alpha\lambda_0}{x_1 x_0} + \frac{1}{\lambda_0 x_0} + 1 &= \frac{\alpha\lambda_0}{x_0\alpha}(x_0 x_{-1} - 1) + \frac{1}{\lambda_0 x_0} + 1 \\ &= \lambda_0 x_{-1} + \frac{\lambda_0}{x_0}\left(-1 + \frac{1}{\lambda_0^2}\right) + 1.\end{aligned}$$

But as λ_0 is a solution of equation (4), we have

$$-1 + \frac{1}{\lambda_0^2} = \alpha\lambda_0.$$

Then

$$\begin{aligned}\frac{\alpha\lambda_0}{x_1 x_0} + \frac{1}{\lambda_0 x_0} + 1 &= \lambda_0 x_{-1} + \frac{\lambda_0^2\alpha}{x_0} + 1 \\ &= \lambda_0 x_{-1}\left(1 + \frac{\lambda_0\alpha}{x_0 x_{-1}}\right) + 1 \\ &= \lambda_0 x_{-1}\left(-\frac{1}{\lambda_0 x_{-1}}\right) + 1 = 0.\end{aligned}$$

This implies that $(x_1, x_0) \in D_1$.

Suppose now that $(x_n, x_{n-1}) \in D_1$. That is

$$\frac{\alpha\lambda_0}{x_n x_{n-1}} + \frac{1}{\lambda_0 x_{n-1}} + 1 = 0.$$

Then

$$\begin{aligned}\frac{\alpha\lambda_0}{x_{n+1} x_n} + \frac{1}{\lambda_0 x_n} + 1 &= \frac{\alpha\lambda_0}{x_n\alpha}(x_n x_{n-1} - 1) + \frac{1}{\lambda_0 x_n} + 1 \\ &= \lambda_0 x_{n-1} + \frac{\lambda_0}{x_n}\left(-1 + \frac{1}{\lambda_0^2}\right) + 1 \\ &= \lambda_0 x_{n-1} + \frac{\lambda_0^2\alpha}{x_n} + 1 \\ &= \lambda_0 x_{n-1}\left(1 + \frac{\lambda_0\alpha}{x_n x_{n-1}}\right) + 1 \\ &= \lambda_0 x_{n-1}\left(-\frac{1}{\lambda_0 x_{n-1}}\right) + 1 = 0.\end{aligned}$$

Therefore, $(x_{n+1}, x_n) \in D_1$ and the proof is complete. □

Definition 3.1 ([30]). A subset $N \subset \mathbb{R}^n$ has lesbegue measure zero if for every $\epsilon > 0$, there exists a collection $\{E_1, E_2, \dots\}$ of Borel sets with

$$N \subset \cup_{i=1}^{\infty} E_i, \quad \sum_{i=1}^{\infty} \mu(E_i) < \epsilon.$$

For more on lesbegue measure, one can see [15, 18, 20, 26, 31].

Theorem 3.2. *The set D_1 is of Lebesgue measure zero.*

Proof. It is required to prove that the set

$$\left\{(x, g(x)) \in \mathbb{R}^2 : g(x) = -\frac{1}{\lambda_0} - \frac{\alpha\lambda_0}{x}\right\}$$

is of Lebesgue measure zero. We can write

$$D_1 = S_+ \cup S_-,$$

where

$$S_+ = \{(x, g(x)) \in \mathbb{R}^2, x > 0\}$$

and

$$S_- = \{(x, g(x)) \in \mathbb{R}^2, x < 0\}.$$

We show that S_+ is of Lebesgue measure zero. For, let

$$S_+ = S_1 \cup S_2,$$

where

$$S_1 = \{(x, g(x)) \in \mathbb{R}^2, 1 \le x < \infty\}$$

and

$$S_2 = \{(x, g(x)) \in \mathbb{R}^2, 0 < x \le 1\}.$$

Now consider the part S_1^n of the set S_1 in the interval $[n, n+1]$. For $\epsilon > 0$, there exists $\delta > 0$ such that $|g(x) - g(y)| < \frac{\epsilon}{2^{n+2}}$ whenever $|x - y| < \delta$ for all $x,\ y \in [n, n+1]$.

Divide the interval $[n, n+1]$ into k subinterval $[n + \frac{i-1}{k}, n + \frac{i}{k}]$, $1 \le i \le k$ with $\frac{1}{k} < \delta$. Then

$$S_1^n \subset \cup_{i=1}^{k} \Big[n + \frac{i-1}{k}, n + \frac{i}{k}\Big] \times \Big[g\big(n + \frac{i-1}{k}\big) - \frac{\epsilon}{2^{n+2}}, g\big(n + \frac{i-1}{k}\big) + \frac{\epsilon}{2^{n+2}}\Big].$$

That is

$$\mu(S_1^n) < \sum_{i=1}^{k} \mu\left(\left[n + \frac{i-1}{k}, n + \frac{i}{k}\right] \times \left[g\big(n + \frac{i-1}{k}\big) - \frac{\epsilon}{2^{n+2}}, g\big(n + \frac{i-1}{k}\big) + \frac{\epsilon}{2^{n+2}}\right]\right)$$
$$= \sum_{i=1}^{k} \frac{1}{k} \frac{\epsilon}{2^{n+1}} = \frac{\epsilon}{2^{n+1}}.$$

But

$$S_1 = \cup_{n=1}^{\infty} S_1^n.$$

Then

$$\mu(S_1) = \sum_{n=1}^{\infty} \mu(S_1^n) < \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} = \frac{\epsilon}{2}.$$

Similarly we can show that $\mu(S_2) < \frac{\epsilon}{2}$. Therefore, $\mu(S_+) < \epsilon$ and S_+ is of Lebesgue measure zero.

The proof that S_- is of Lebesgue measure zero is similar and will be omitted. Thus $D_1 = S_+ \cup S_-$ is a union of two Lebesgue measure zero subsets is also of Lebesgue measure zero.

This completes the proof. □

In case $\alpha = \frac{2}{3\sqrt{3}}$, consider the following subsets:

$$D_1 = \left\{(x,y) \in \mathbb{R}^2 : \frac{1}{3xy} + \frac{2}{\sqrt{3}y} + 1 = 0\right\},$$
$$D_2 = \left\{(x,y) \in \mathbb{R}^2 : \frac{8}{3xy} - \frac{2}{\sqrt{3}y} - 1 = 0\right\},$$
$$D_3 = \left\{(x,y) \in \mathbb{R}^2 : \frac{2}{3xy} + \frac{1}{\sqrt{3}y} - 1 = 0\right\}.$$

Note that D_1, D_2 and D_3 are equivalent to $c_1(x,y) = 0$, $c_2(x,y) = 0$ and $c_3(x,y) = 0$ respectively. The two subsets D_1 and D_3 are invariant subsets for equation (2).

Finally when $\alpha < \frac{2}{3\sqrt{3}}$. We shall consider the three sets

$$D_i = \{(x,y) \in \mathbb{R}^2 : \frac{\alpha\lambda}{xy} + \frac{1}{\lambda y} + 1 = 0\}, \quad i = 1, 2, 3,$$

where

$$\begin{cases} \lambda = \lambda_0, & i = 1; \\ \lambda = \lambda_-, & i = 2; \\ \lambda = \lambda_+, & i = 3. \end{cases}$$

By simple calculations, we can see that:

$$\begin{cases} D_i \text{ is equivalent to } & c_1(x,y) = 0, \quad i = 1; \\ D_i \text{ is equivalent to } & c_2(x,y) = 0, \quad i = 2; \\ D_i \text{ is equivalent to } & c_1(x,y) = 0, \quad i = 3. \end{cases}$$

Theorem 3.3. *Each set of the sets D_i, $i = 1$, 2 and 3 is an invariant for equation* (2).

Proof. The proof is similar to that of theorem (3.1) and will be omitted. □

4. GLOBAL BEHAVIOR OF EQUATION (2)

In this section, we shall investigate the behavior of equation (2) for all values of α.

In the following results, we show that when $\alpha > \frac{2}{3\sqrt{3}}$, under certain conditions there exist solutions, that are either periodic or converging to periodic solutions for equation (2).

Suppose that $\theta = \frac{p}{q}\pi$ is a rational multiple of π, where p and q are positive relatively prime integers such that $\frac{q}{2} < p < q$.

Theorem 4.1. *Assume that* $\alpha > \frac{2}{3\sqrt{3}}$ *and let* $(x_0, x_{-1}) \notin F$. *Then we have the following:*

1. *If* $(x_0, x_{-1}) \notin D_1$, *then* $\{x_n\}_{n=-1}^{\infty}$ *converges to a periodic solution with prime period* q.
2. *If* $(x_0, x_{-1}) \in D_1$, *then* $\{x_n\}_{n=-1}^{\infty}$ *is a periodic solution with prime period* q.

Proof. When $\alpha > \frac{2}{3\sqrt{3}}$, the solution of equation (2) can be written

$$x_{qm+i} = \frac{c_1\lambda_0^{qm+i-1} + (\frac{1}{\alpha\lambda_0})^{\frac{qm+i-1}{2}}(c_2\cos(qm+i-1)\theta + c_3\sin(qm+i-1)\theta)}{c_1\lambda_0^{qm+i} + (\frac{1}{\alpha\lambda_0})^{\frac{qm+i}{2}}(c_2\cos(qm+i)\theta + c_3\sin(qm+i)\theta)},$$

with $1 \le i \le q$.

(1) For each $i = 1, 2, \dots, q$, we get

$$x_{qm+i} = \sqrt{\lambda_0\alpha}\frac{c_1(\lambda_0^3\alpha)^{\frac{qm+i-1}{2}} + c_2\cos(pm\pi + (i-1)\theta) + c_3\sin(pm\pi + (i-1)\theta)}{c_1(\lambda_0^3\alpha)^{\frac{qm+i}{2}} + c_2\cos(pm\pi + i\theta) + c_3\sin(pm\pi + i\theta)}.$$

As $m \to \infty$, we get

$$x_{qm+i} \to \sqrt{\lambda_0\alpha}\frac{c_2\cos(i-1)\theta + c_3\sin(i-1)\theta}{c_2\cos i\theta + c_3\sin i\theta}.$$

(2) Assume that $(x_0, x_{-1}) \in D_1$. Then for each $i = 1, 2, \dots, q$, we get

$$\begin{aligned} x_{qm+i} &= \sqrt{\lambda_0\alpha}\frac{c_2\cos(qm+i-1)\theta + c_3\sin(qm+i-1)\theta}{c_2\cos(qm+i)\theta + c_3\sin(qm+i)\theta} \\ &= \sqrt{\lambda_0\alpha}\frac{c_2\cos(pm\pi + (i-1)\theta) + c_3\sin(pm\pi + (i-1)\theta)}{c_2\cos(pm\pi + i\theta) + c_3\sin(pm\pi + i\theta)} \\ &= \sqrt{\lambda_0\alpha}\frac{c_2\cos(i-1)\theta + c_3\sin(i-1)\theta}{c_2\cos i\theta + c_3\sin i\theta}. \end{aligned}$$ □

Suppose that $\theta \in \mathbb{R} - \pi\mathbb{Q}$ is not a rational multiple of π.

Theorem 4.2. *Assume that* $\alpha > \frac{2}{3\sqrt{3}}$. *If* $\theta \in \mathbb{R} - \pi\mathbb{Q}$ *is not a rational multiple of* π, *then the solution* $\{x_n\}_{n=-1}^{\infty}$ *is dense in* $\mathbb{R}$.

Proof. We can write the solution (5) of equation (2) as

$$x_n = \sqrt{\lambda_0 \alpha}\frac{c_1(\lambda_0^3\alpha)^{\frac{n-1}{2}} + A\sin((n-1)\theta + \varphi)}{c_1(\lambda_0^3\alpha)^{\frac{n}{2}} + A\sin(n\theta + \varphi)},$$

where $A = \sqrt{c_2^2 + c_3^2}$ and $\varphi = \tan^{-1}\frac{c_2}{c_3} \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$. Since $\frac{\theta}{\pi}$ is an irrational number, we can find for each $l \in \mathbb{R}$ a sequence $w_k = n_k\theta + \varphi - 2\pi m_k$, where $\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ are sequences of positive integers such that

$$\lim_{k\to\infty} w_k = l.$$

Then

$$x_{n_k} = \sqrt{\lambda_0 \alpha}\frac{c_1(\lambda_0^3\alpha)^{\frac{n_k-1}{2}} + A\sin(w_k - \theta)}{c_1(\lambda_0^3\alpha)^{\frac{n_k}{2}} + A\sin w_k}.$$

As $k \to \infty$, we get

$$x_{n_k} \to \sqrt{\lambda_0\alpha}\frac{\sin(l-\theta)}{\sin l}.$$

Now, consider the function

$$\begin{aligned} &f : \mathbb{R}/\pi\mathbb{Z} \to \mathbb{R}, \\ &x \mapsto \sqrt{\lambda_0\alpha}\frac{\sin(t-\theta)}{\sin t}, \text{ where } t \notin \pi\mathbb{Z}. \end{aligned}$$

As the function is surjective, each number $z \in \mathbb{R}$ can be written as $\sqrt{\lambda_0\alpha}\frac{\sin(t-\theta)}{\sin t}$ for some $t \in \mathbb{R}$. Then each number $\sqrt{\lambda_0\alpha}\frac{\sin(t-\theta)}{\sin t} \in \mathbb{R}$ is a limit point of a sequence of the set $\{x_n : n \geq -1\}$.

This completes the proof. □

Now returning to equation (2), where the equilibrium points depends on the values of γ. The equilibrium points are classified as the following:

- If $\alpha > \frac{2}{3\sqrt{3}}$, then there is a unique positive equilibrium point $\bar{r}_1 > \frac{2}{\sqrt{3}}$.
- If $\alpha = \frac{2}{3\sqrt{3}}$, then there are two equilibrium points, $\bar{r}_1 = \frac{2}{\sqrt{3}}$ and a negative equilibrium point $\bar{r}_2 = -\frac{1}{\sqrt{3}}$.
- If $\alpha < \frac{2}{3\sqrt{3}}$, then there are three equilibrium points, $\bar{r}_1 < \frac{2}{\sqrt{3}}$ and two negative equilibrium points $\bar{r}_2$ and $\bar{r}_3$ such that

$$\begin{aligned} \bar{r}_2 &= -\frac{\bar{r}_1}{2} + \frac{\sqrt{-3\bar{r}_1^2 + 4}}{2}, \\ \bar{r}_3 &= -\frac{\bar{r}_1}{2} - \frac{\sqrt{-3\bar{r}_1^2 + 4}}{2} \end{aligned}$$

and

$$\bar{r}_3 < -\frac{1}{\sqrt{3}} < \bar{r}_2 < 0.$$

Theorem 4.3. *Assume that* $\alpha = \frac{2}{3\sqrt{3}}$ *and let* $(x_0, x_{-1}) \notin F$. *If* $(x_0, x_{-1}) \notin \cup_{i=1}^{3} D_i$, *then the solution* $\{x_n\}_{n=-1}^{\infty}$ *converges to the negative equilibrium point* $\bar{r}_2 = -\frac{1}{\sqrt{3}}$.

Proof. We have that

$$\begin{aligned} x_n &= \frac{c_1(\frac{1}{3\alpha})^{n-1} + c_2(-\frac{2}{3\alpha})^{n-1} + c_3(-\frac{2}{3\alpha})^{n-1}(n-1)}{c_1(\frac{1}{3\alpha})^{n} + c_2(-\frac{2}{3\alpha})^{n} + c_3(-\frac{2}{3\alpha})^{n}n} \\ &= \frac{c_1(\frac{\sqrt{3}}{2})^{n-1} + c_2(-\sqrt{3})^{n-1} + c_3(-\sqrt{3})^{n-1}(n-1)}{c_1(\frac{\sqrt{3}}{2})^{n} + c_2(-\sqrt{3})^{n} + c_3(-\sqrt{3})^{n}n} \\ &= \frac{1}{\sqrt{3}} \frac{c_1(\frac{1}{2})^{n-1} + c_2(-1)^{n-1} + c_3(-1)^{n-1}(n-1)}{c_1(\frac{1}{2})^{n} + c_2(-1)^{n} + c_3(-1)^{n}n}. \end{aligned}$$

Clear that x_{2n} and x_{2n+1} converge to $-\frac{1}{\sqrt{3}}$, from which the result follows. □

Note that: When $\alpha = \frac{2}{3\sqrt{3}}$, the invariant subsets D_1 and D_3 intersect at $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$.

Theorem 4.4. *Assume that* $\alpha < \frac{2}{3\sqrt{3}}$. *Then the equilibrium point* $\bar{r_2}$ *attracts all orbits with initial points outside a set of Lebesgue measure zero.*

Proof. Suppose that $(x_0, x_{-1}) \notin F \cup (\cup_{i=1}^{3} D_i)$. The set of Lebesgue measure zero is the set $F \cup (\cup_{i=1}^{3} D_i)$. In fact it is a union of Lebesgue measure zero subsets of $\mathbb{R}^2$.

The solution of equation (2) is

$$\begin{aligned} x_n &= \frac{c_1\lambda_0^{n-1} + c_2\lambda_-^{n-1} + c_3\lambda_+^{n-1}}{c_1\lambda_0^{n} + c_2\lambda_-^{n} + c_3\lambda_+^{n}} \\ &= \frac{1}{\lambda_-} \frac{c_1(\frac{\lambda_0}{\lambda_-})^{n-1} + c_2 + c_3(\frac{\lambda_+}{\lambda_-})^{n-1}}{c_1(\frac{\lambda_0}{\lambda_-})^{n} + c_2 + c_3(\frac{\lambda_+}{\lambda_-})^{n}}. \end{aligned}$$

As $n \to \infty$,

$$x_n \to \frac{1}{\lambda_-} = \bar{r}_2.$$

□

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